MATH2040A/B Homework 8 Solution

1. (f) For $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$, $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-1 & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda)$.
   Hence the eigenvalues of A are $\lambda = 1$ or 3. For $\lambda = 1$,
   $$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{\text{RREF}}{\longrightarrow} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
   then $E_\lambda = \operatorname{span}\{(1,0,0)^T\}$. Since $\gamma_A(1) = 1, \mu_A(1) = 2$, A is not diagonalizable.

2. Denote $\beta' = \{1, x, x^2\}$ as the standard ordered basis of $P_2(\mathbb{R})$. Then we have
   $$[T]_{\beta'} = \left([T(1)]_{\beta'} \quad [T(x)]_{\beta'} \quad \left[T\left(x^2\right)\right]_{\beta'}\right) = \left(\left[x^2\right]_{\beta'} \quad [x]_{\beta'} \quad [1]_{\beta'}\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
   We proceeds to find the characteristic function of T, given by
   $$f_T(t) = \det\left([T]_{\beta'} - tI_3\right) = \det\begin{pmatrix} -t & 0 & 1 \\ 0 & 1-t & 0 \\ 1 & 0 & -t \end{pmatrix} = (1-t)\det\begin{pmatrix} -t & 1 \\ 1 & -t \end{pmatrix} = -(1-t)^2(t+1)$$
   The characteristic polynomial of T splits. It follows that the root is given by $\lambda_1 = 1$ and $\lambda_2 = -2$, and their algebraic multiplicity is given by 2 and 1 respectively. We then find the eigenspaces one by one.
   – $(\lambda_1 = 1)$ The eigenspace associated with the eigenvalue $\lambda = \lambda_1 = 1$ is given by
   $$E_{\lambda_1} = N\left([T]_{\beta'} - I_3\right) = N\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \left\{ t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : t, s \in \mathbb{R} \right\}$$
   Therefore, $\mu_T(1) = 2 = \dim\left(E_{\lambda_1}\right) =: \gamma_T(1)$
   – $(\lambda_2 = -1)$ The eigenspace associated with the eigenvalue $\lambda = \lambda_2 = -1$ is given by
   $$E_{\lambda_2} = N\left([T]_{\beta'} + I_3\right) = M\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \left\{ t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$$
   Therefore, $\mu_T(-1) = 1 = \dim\left(E_{\lambda_2}\right) =: \gamma_T(-1)$
   As $\mu_T(1) = \gamma_T(1)$ and $\mu_T(-1) = \gamma_T(-1)$, it follows that T is diagonalizable. As T is diagonalizable, also $\beta_1 := \left\{(1,0,1)^T, (0,1,0)^T\right\}$ and $\beta_2 := \left\{(1,0,-1)^T\right\}$ are the ordered basis of E_{λ_1} and E_{λ_2} respectively. It follows that $\beta := \beta_1 \cup \beta_2 = \left\{(1,0,1)^T, (0,1,0)^T, (1,0,-1)^T\right\}$ is an ordered basis for V consisting of eigenvectors and hence
   $$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3. Diagonalize the matrix A by $Q^{-1}AQ = D$ with $D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$.
   So we know that
   $$A^n = QD^nQ^{-1} = Q\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix} Q^{-1}.$$

4. As $[T]_\beta$ is upper triangular, denote the i -th diagonal entries as δ_i, then we have

$$f_T(t) = \det([T]_\beta - tI) = \prod_{i=1}^{n} (\delta_i - t) = 0$$

have the roots $\delta_1, \cdots, \delta_n$, where $n = \dim(V)$. As proven in the previous assignment, $[T]_\beta$ and T has same set of eigenvalues. Therefore, $\delta_1, \cdots, \delta_n$ is the set of eigenvalues for T and hence it must be the case that λ_i appears m_i times exactly within $\delta_1, \cdots, \delta_n$. The statement then holds.

5. (a) We may pick one basis α such that both $[T]_\alpha$ and $[U]_\alpha$ are diagonal. Let $Q = [I]_\alpha^\beta$. And we will find out that

$$[T]_\alpha = Q^{-1}[T]_\beta Q$$

and

$$[U]_\alpha = Q^{-1}[U]_\beta Q.$$

(b)Let Q be the invertible matrix who makes A and B simultaneously diagonalizable. Say β be the basis consisting of the column vectors of Q. And let α be the standard basis. Now we know that

$$[T]_\beta = [I]_\alpha^\beta [T]_\alpha [I]_\beta^\alpha = Q^{-1}AQ$$

and

$$[U]_\beta = [I]_\alpha^\beta [U]_\alpha [I]_\beta^\alpha = Q^{-1}BQ.$$

6. (e) No. For $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in W$, we have

$$T(A) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \notin W.$$

7. Let $\{W_i\}_{i \in I}$ be the collection of T -invatirant subspaces and W be the intersection of them. For every $v \in W$, we have $T(v) \in W_i$ for every $i \in I$, since v is an element is each W_i. This means $T(v)$ is also an element in W.

8. (a) Let w be an element in W. We may express w to be

$$w = \sum_{i=0}^{k} a_i T^i(v)$$

And thus we have

$$T(w) = \sum_{i=0}^{k} a_i T^{i+1}(v) \in W.$$

(b) Let U be a T -invariant subspace of V containing v. since it's T invariant, we know that $T(v)$ is an element in U. Inductively, we know that $T^k(v) \in U$ for all nonnegative integer k. By Theorem 1.5 we know that U must contain W.

Remark: Theorem 1.5: The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S

9. If w is an element in W, it's a linear combination of

$$\{v, T(v), T^2(v), \ldots$$

So $w = g(T)(v)$ for some polynomial g. Conversely, if $w = g(T)(v)$ for some polynomial g, this means w is a linear combination of the same set. Hence w is an element in W.

10. Define

$$A_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix} \Rightarrow A_k - tI_k = \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} - t \end{pmatrix}$$

Let $P(n)$ be the statement that the characteristic polynomial of A_n is given by

$$(-1)^n \left(a_0 + \cdots + a_{n-1}t^{n-1} + t^n\right).$$

For $n = 1$, notice that $A_1 = -a_0 - t = (-1)^1 \left(a_0 + t^1\right)$. $P(1)$ is true. Suppose $P(k)$ is true for some $k \in \mathbb{N}$, i.e.

$$\det\left(A_k - tI_k\right) = \det \begin{pmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} - t \end{pmatrix} = (-1)^k \left(a_0 + \cdots + a_{k-1}t^{k-1} + t^k\right)$$

It follows that

$$\begin{aligned} \det\left(A_{k+1} - tI_{k+1}\right) &= \det \begin{pmatrix} -t & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & -t & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -t & -a_{k-1} \\ 0 & 0 & \cdots & 0 & 1 & -a_k - t \end{pmatrix}, \\ &= \det \begin{pmatrix} -t & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & -t & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -t & -a_{k-1} - t \\ 0 & 0 & \cdots & 0 & 1 & -a_k - t + 1 \end{pmatrix} \end{aligned}$$

where the last equality follows from adding the second last column to the last column. Consider expanding the above determinant along the last row, we obtained that

$$\begin{aligned} \det\left(A_{k+1} - tI_{k+1}\right) &= -\det\left(A_k - tI_k\right) + \left(-a_k - t + 1\right) \det \begin{pmatrix} -t & 0 & \cdots & 0 & 0 \\ 1 & -t & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 0 \\ 0 & 0 & \cdots & 1 & -t \end{pmatrix}. \\ &= -(-1)^k \left(a_0 + \cdots + a_{k-1}t^{k-1} + t^k\right) + \left(-a_k - t + 1\right)(-t)^k \\ &= (-1)^{k+1} \left(a_0 + \cdots + a_{k-1}t^{k-1} + t^k\right) + (-1)^{k+1} \left(a_k t^k + t^{k+1} - t^k\right) \\ &= (-1)^{k+1} \left(a_0 + \cdots + a_{k-1}t^{k-1} + a_k t^k + t^{k+1}\right) \end{aligned}$$

Therefore, $P(k+1)$ also holds. It follows by principle of mathematical induction that the characteristic polynomial of A is $(-1)^k \left(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k\right)$.